The ArcLength Parametrization of a Curve

Assume we are given a space curve C determined by the vector function $\vec{r}(t) = \; < f(t), g(t), h(t) >$ were $t \in [\,a, b]$. Let L denote the total length of C. We know $L = \displaystyle\int_a^b \left|\left|\vec{r}\,'(u)\right|\right| du.$

Now we are going to re-parameterize C using a very special parameter s called the arclength parameter. First, $s \in [\,0, L]$. Second, and more important, for each such s there exists a unique point P such that s gives the length of the curve starting from the point $A(f(a), g(a), h(a))$ and stopping at P. Thus we can imagine that the xyz coordinates of P are all functions of s. We can assume the same curve C may be parametrized by another vector function $\vec{\alpha}(s) = \; < F(s), G(s), H(s) >$ where $s \in [\,0, L]$. Such an imagined parametrization is called the arclength parametrization of C where the variable s is called the arclength parameter. The letter α is from the Greek alphabet. In this context, α represents arclength.

We know that the curve C starts at the point A whose coordinates can now be thought of in two ways as either $A(f(a), g(a), h(a))$ when we think of the t parameter, or the starting point A may be thought of as being $A(F(0), G(0), H(0))$ when we think in terms of the arclength parameter s and set $s = 0$. We also know C ends at the point B whose coordinates are both $B(f(b), g(b), h(b))$ and $B(F(L), G(L), H(L))$ where the last set of coordinates can be thought of if we set $s = L$.

More generally, for each $t \in [\,a, b]$ we can define a length function $l(t) = \displaystyle\int_a^t \left|\left|\vec{r}\,'(u)\right|\right| du$. This length function maps the closed interval $[\,a, b]$ onto the closed interval $[\,0, L]$ and it does so in a 1-1 fashion because $l(t)$ is a continuous increasing function. Thus we can think that for each $t \in [\,a, b]$ the arclength $s = l(t)$ and that $t = l^{-1}(s)$ and each point on the curve that we originally thought of as being given by the position vector endpoint $< f(t), g(t), h(t) >$ can now be thought of in terms of another vector function endpoint $\vec{\alpha}(s) = \; < f(l^{-1}(s)),\, g(l^{-1}(s)),\, h(l^{-1}(s)) >$. In other words, armed with the function $s = l(t)$ we discover that $F(s) = f(l^{-1}(s)),\; G(s) = g(l^{-1}(s)),$ and $H(s) = h(l^{-1}(s)$ and these equations tell us how to theoretically determine the arclength parametric components of $\vec{\alpha}(s)$. In practice, it may be impossible to write an explicit formula for the $l^{-1}(s)$ function, even when the curve C is a very simple curve. A simple 2D example will illustrate this peculiar property of the arclength parametrization. Consider the parabola given by:

$$\vec{r}(t) = \; < t, 2 - t^2, 0 > \text{ where } t \in [\,0,\ \sqrt{2}\,].$$

$$l(t) = \int_0^t \sqrt{1 + (-2u)^2}\, du = \int_0^t \sqrt{1 + 4u^2}\, du = u\sqrt{\frac{1}{4} + u^2} + \frac{1}{4} ln\left(u + \sqrt{\frac{1}{4} + {}^2}\right)\Bigg|_0^t$$

$$l(t) = t\sqrt{\frac{1}{4} + t^2} + \frac{1}{4} ln\left(t + \sqrt{\frac{1}{4} + t^2}\right) - ln\left(\sqrt[4]{\frac{1}{2}}\right).$$

No matter how good your high school algebra is, you should find it impossible to derive an explicit formula for $l^{-1}(t)$. To do so would require solving $s = l(t)$ for t in terms of s.

Thus when we speak about the functions $F(s),\ G(s),$ and $H(s)$ for the arclength parametrization, we can almost never write actual formulas for these functions. And if we can't write the formulas, we can't actually do any calculations with these parametric functions either. So what is the arclength parametrization $\vec{\alpha}(s)$ good for? It is good for thinking about the curve C and has some very simple but ultimately useful properties. The first property is that the vector function $\vec{\alpha}'(s)$ not only has constant length, it actually has unit length! The reason for this is that

$$s = \int_0^s \left|\left| \vec{\alpha}'(u) \right|\right| du$$

Now if we differentiate both sides of this equation with respect to s we get

$$1 = \left|\left| \vec{\alpha}'(s) \right|\right|$$

Then we can square both sides of this last equation and use the fact that the length of a vector squared is the same as the dot product of the vector with itself.

$$1 = \left|\left| \vec{\alpha}'(s) \right|\right|^2 = \ < F'(s), G'(s), H'(s) > \ \cdot \ < F'(s), G'(s), H'(s) >$$

$$1 = [F'(s)]^2 + [G'(s)]^2 + [H'(s)]^2$$

Now we again differentiate both sides of the equation with respect to s :

$$0 = 2 \cdot F'(s) \cdot F''(s) + 2 \cdot G'(s) \cdot G''(s) + 2 \cdot H'(s) \cdot H''(s)$$

and after dividing by 2

$$0 = F'(s) \cdot F''(s) + G'(s) \cdot G''(s) + H'(s) \cdot H''(s)$$

$$0 = \ < F'(s), G'(s), H'(s) > \ \cdot \ < F''(s), G''(s), H''(s) > \ = \ \vec{\alpha}'(s) \cdot \vec{\alpha}''(s)$$

We have shown $\vec{\alpha}'(s) \cdot \vec{\alpha}''(s) = 0$ so that the derivative of the vector function $\vec{\alpha}'(s)$ is orthogonal to this same vector function itself.

Now let's keep in mind that the arclength parameter s isn't necessarily time, so that the two vectors $\vec{\alpha}'(s)$ and $\vec{\alpha}''(s)$ are not necessarily velocity and acceleration vectors. But how do they relate to velocity and acceleration and the time variable t?

First, $\vec{\alpha}'(s)$ is a unit tangent vector. But $\vec{r}'(t)$ is a tangent vector whose length is $\left|\left|\vec{r}'(t)\right|\right|$. Because any tangent vector must point in the same direction as the increase in the parameter, and since t and s increase in the same direction along C, we may start with the unit tangent vector $\vec{\alpha}'(s)$ and stretch it just the correct amount to make the real tangent or velocity vector $\vec{r}'(t)$. In other words,

$$\left|\left|\vec{r}'(t)\right|\right| \cdot \vec{\alpha}'(s) = \vec{r}'(t)$$

Another way of saying the same thing is that:

$$\vec{\alpha}'(s) = \frac{1}{\left|\left|\vec{r}'(t)\right|\right|} \cdot \vec{r}'(t)$$

By the arclength parametrization we know

$$s = \int_a^t \left|\left|\vec{r}'(u)\right|\right| du$$

Now differentiate both sides of this equation with respect to t.

$$\frac{ds}{dt} = \left|\left|\vec{r}'(t)\right|\right|$$

By the Chain Rule,

$$\frac{d\,\vec{r}(t)}{ds} = \frac{d\,\vec{r}(t)}{dt} \cdot \frac{dt}{ds} = \frac{\frac{d\,\vec{r}(t)}{dt}}{\frac{ds}{dt}} = \frac{\vec{r}'(t)}{\left|\left|\vec{r}'(t)\right|\right|} = \vec{\alpha}'(s)$$

This shows that the unit tangent vector $\vec{\alpha}'(s)$ is the derivative or rate of change of $\vec{r}(t)$ with respect to the arclength variable s. This is another reason why the arclength parameter s is so important.

There is another rate of change with respect to arclength that we will define as the **curvature vector**. Formally, we define the curvature vector to be the vector $\vec{\alpha}''(s)$. This is the rate of change of the unit tangent vector with respect to arclength s.

Finally, we define curvature as the length of the curvature vector. We will denote the curvature scalar quantity by the letter k and we may write:

$$k = \left|\left|\vec{\alpha}''(s)\right|\right|$$

Now our final problem is to calculate k using the function formulas found inside the original vector function $\vec{r}(t)$ that was used to define C in the first place. In other words, since we will almost never have explicit formulas for $F(s)$, $G(s)$ or $H(s)$ we desperately need to calculate k in terms of $f(t)$, $g(t)$, and $h(t)$.

$$\vec{r}'(t) = \left|\left|\vec{r}'(t)\right|\right| \cdot \vec{\alpha}'(s)$$

$$\vec{r}'(t) = \frac{ds}{dt} \cdot \vec{\alpha}'(s)$$

Differentiate both sides of the last equation above, implicitly, with respect to t :

$$\vec{r}''(t) = \frac{d^2s}{dt^2} \cdot \vec{\alpha}'(s) + \frac{ds}{dt} \cdot \frac{d\vec{\alpha}'(s)}{dt}$$

$$\vec{r}''(t) = \frac{d^2s}{dt^2} \cdot \vec{\alpha}'(s) + \frac{ds}{dt} \cdot \frac{d\vec{\alpha}'(s)}{ds} \cdot \frac{ds}{dt}$$

$$\vec{r}''(t) = \frac{d^2s}{dt^2} \cdot \vec{\alpha}'(s) + \left(\frac{ds}{dt}\right)^2 \vec{\alpha}''(s)$$

Now take the cross product with $\vec{r}'(t)$ on both sides of the last equation above.

$$\vec{r}'(t) \times \vec{r}''(t) = \vec{r}'(t) \times \left[\frac{d^2s}{dt^2} \cdot \vec{\alpha}'(s) + \left(\frac{ds}{dt}\right)^2 \cdot \vec{\alpha}''(s)\right]$$

(the first vector inside the square brackets is already parallel to $\vec{r}'(t)$, so the first term drops out)

$$\vec{r}'(t) \times \vec{r}''(t) = \vec{r}'(t) \times \left[\left(\frac{ds}{dt}\right)^2 \cdot \vec{\alpha}''(s)\right]$$

$$\vec{r}'(t) \times \vec{r}''(t) = \left(\frac{ds}{dt}\right)^2 \cdot \left[\vec{r}'(t) \times \vec{\alpha}''(s)\right]$$

Now take the length of the vectors on both sides of this last equation:

$$\left|\left|\vec{r}'(t) \times \vec{r}''(t)\right|\right| = \left(\frac{ds}{dt}\right)^2 \cdot \left|\left|\vec{r}'(t)\right|\right| \cdot \left|\left|\vec{\alpha}''(s)\right|\right| \cdot |sin(\theta)|$$

where θ is the angle between the two vectors $\vec{r}'(t)$ and $\vec{\alpha}''(s)$. After noting this angle is 90° and $sin(90^{\circ}) = 1$ and noting that $\frac{ds}{dt} = \left|\left|\vec{r}'(t)\right|\right|$ we may write:

$$\left|\left|\vec{r}'(t) \times \vec{r}''(t)\right|\right| = \left|\left|\vec{r}'(t)\right|\right|^3 \cdot \left|\left|\vec{\alpha}''(s)\right|\right|$$

Solving for the curvature, the final equation is:

$$k = \left|\left|\vec{\alpha}''(s)\right|\right| = \frac{\left|\left|\vec{r}'(t) \times \vec{r}''(t)\right|\right|}{\left|\left|\vec{r}'(t)\right|\right|^3}$$

When the curvature k isn't zero we define $\rho = \dfrac{1}{k}$ as the radius of curvature. We also define the osculating plane to be the plane determined by the tangent vector $\vec{\alpha}'(s)$ and the curvature vector $\vec{\alpha}''(s)$.

We are going to formally define the unit tangent vector by writing:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\left|\left|\vec{r}'(t)\right|\right|}$$

Since this vector has constant length, by a previous theorem it is orthogonal to its own derivative vector which is $\vec{T}'(t)$. Now the vector $\vec{T}'(t)$ isn't necessarily a unit vector, but we can make a unit vector in the same direction as $\vec{T}'(t)$ by defining a new vector :

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left|\left|\vec{T}'(t)\right|\right|}$$

Then we know $\vec{T}(t)$ and $\vec{N}(t)$ are unit vectors that are orthogonal to each other. $\vec{N}(t)$ is called the normal vector to the curve C. Incidentally, $\vec{T}(t) = \vec{\alpha}'(s)$ and $\vec{N}(t) = \dfrac{\vec{\alpha}''(s)}{\left|\left|\vec{\alpha}''(s)\right|\right|}$.

So $\vec{T}$ is the unit tangent vector and $\vec{N}$ is the unit normal vector. The plane that contains $\vec{T}$ and $\vec{N}$ is called the osculating plane. The plane for which $\vec{T}$ serves as a normal vector is called the normal plane. A third unit vector is called the binormal vector which we denote by $\vec{B}$. We define $\vec{B} = \vec{T} \times \vec{N}$. The plane that contains $\vec{B}$ and $\vec{T}$ is called the rectifying plane. The normal plane contains the $\vec{B}$ and $\vec{N}$ vectors. Like $\vec{T}$ and $\vec{N}$, $\vec{B}$ itself must be a unit vector because by the property of cross products,

$\left|\left|\vec{B}\right|\right| = \left|\left|\vec{T} \times \vec{N}\right|\right| = \left|\left|\vec{T}\right|\right| \cdot \left|\left|\vec{N}\right|\right| \cdot sin(\theta) = 1 \cdot 1 \cdot 1 = 1$ where $\theta = \dfrac{\pi}{2}$ is the angle between the two orthogonal vectors $\vec{T}$ and $\vec{N}$.

Like curvature, we would like to compute the $\vec{T}$, $\vec{N}$, and $\vec{B}$ vectors in terms of the original vector function $\vec{r}(t)$. The unit tangent vector has already been done.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\left|\left|\vec{r}'(t)\right|\right|}$$

The formula $\vec{r}\,'(t) = \left|\left|\vec{r}\,'(t)\right|\right| \cdot \vec{T}(t)$ can be written more simply as $\vec{r}\,'(t) = v \cdot \vec{T}(t)$ where we let v denote the scalar speed. $\left|\left|\vec{r}\,'(t)\right|\right| = v = \dfrac{ds}{dt} =$ the rate of change of arclength with respect to time. We are using three different notations for the same quantity. Now differentiate both sides of the equation $\vec{r}\,'(t) = v \cdot \vec{T}(t)$ with respect to t.

$$\vec{r}\,''(t) = \frac{dv}{dt} \cdot \vec{T}(t) + v \cdot \vec{T}\,'(t)$$

$$\vec{r}\,''(t) = \frac{dv}{dt} \cdot \vec{T}(t) + v \cdot \left|\left|\vec{T}\,'(t)\right|\right| \cdot \vec{N}(t)$$

We claim that $\left|\left|\vec{T}\,'(t)\right|\right| = kv$ so that the last equation written above simplifies to

$$\vec{r}\,''(t) = \frac{dv}{dt} \cdot \vec{T}(t) + kv^2 \cdot \vec{N}(t)$$

$$\left|\left|\vec{T}\,'(t)\right|\right| = \left|\left|\frac{d\,\vec{T}(t)}{dt}\right|\right| = \left|\left|\frac{d\,\vec{T}(t)}{ds} \cdot \frac{ds}{dt}\right|\right| = \left|\left|\frac{d\,\vec{\alpha}\,'(s)}{ds} \cdot \frac{ds}{dt}\right|\right| = \left|\left|\vec{\alpha}\,''(s) \cdot \frac{ds}{dt}\right|\right| = \left|\left|\vec{\alpha}\,''(s) \cdot v\right|\right|$$

$$= v \cdot \left|\left|\vec{\alpha}\,''(s)\right|\right| = vk.$$

The equation $\vec{r}\,''(t) = \dfrac{dv}{dt} \cdot \vec{T}(t) + kv^2 \cdot \vec{N}(t)$ implies that the tangential and normal components of acceleration are $\dfrac{dv}{dt}$ and kv^2 respectively. Finally, the following two equations allow us to actually calculate the components of the acceleration vector with respect to the tangential and normal vectors $\vec{T}(t)$ and $\vec{N}(t)$. We can do these calculations in terms of the original parametrization of C with the components of the vector function $\vec{r}(t)$.

$$\text{the tangential component } = \frac{dv}{dt} = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{\left|\left|\vec{r}\,'(t)\right|\right|}$$

$$\text{the normal component} = kv^2 = \frac{\left|\left|\vec{r}\,'(t) \times \vec{r}\,''(t)\right|\right|}{\left|\left|\vec{r}\,'(t)\right|\right|}$$

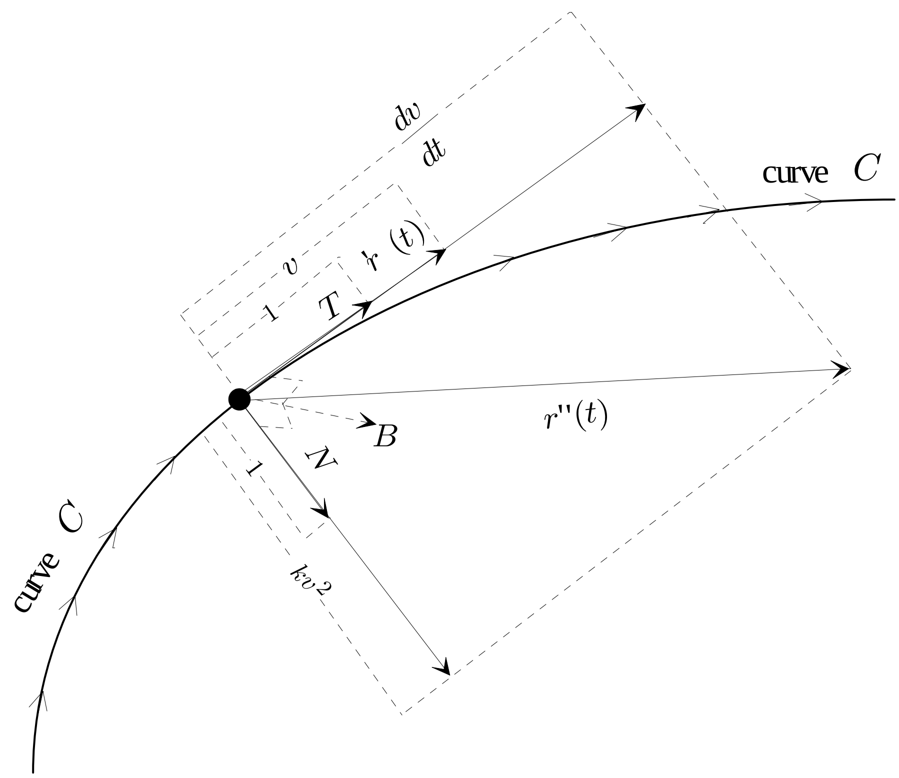
The justification for computing $\frac{dv}{dt}$ using the dot product of $\vec{r}\,'(t)$ with $\vec{r}\,''(t)$ is that $v = \left|\left|\vec{r}\,'(t)\right|\right| =$

$\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$ so we may compute the derivative of v with respect to t directly as :

$$\frac{dv}{dt} = \frac{1}{2}\Big([f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2\Big)^{-\frac{1}{2}} \cdot [2f'(t)\cdot f''(t) + 2g'(t)\cdot g''(t) + 2h'(t)\cdot h''(t)]$$

which immediately simplifies to : $\dfrac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{\left|\left|\vec{r}\,'(t)\right|\right|}$

The following figure shows the relationships between the $\vec{T}, \vec{N},$ and $\vec{B}$ vectors and the tangential and normal components of acceleration.



We should note that since $\vec{T}$and $\vec{N}$ are orthogonal to each other, the parallelogram for adding the two vectors $\frac{dv}{dt}\cdot\vec{T}(t) + kv^2\cdot\vec{N}(t)$ is really a rectangle which has $\vec{r}\,''(t)$ as its diagonal. A practical consequence is that if we know two of the three sides in one of the half-triangles then we can get the 3rd side for free! We may not always need to separately calculate $\frac{dv}{dt}$ and kv^2. We need only calculate two of the following three quantities: $\frac{dv}{dt}$ or kv^2 or $\left|\left|\vec{r}\,''(t)\right|\right|$ and apply the Pythagorean Theorem to get the third. $(kv^2)^2 + \left(\frac{dv}{dt}\right)^2 = \left|\left|\vec{r}\,''(t)\right|\right|^2$.

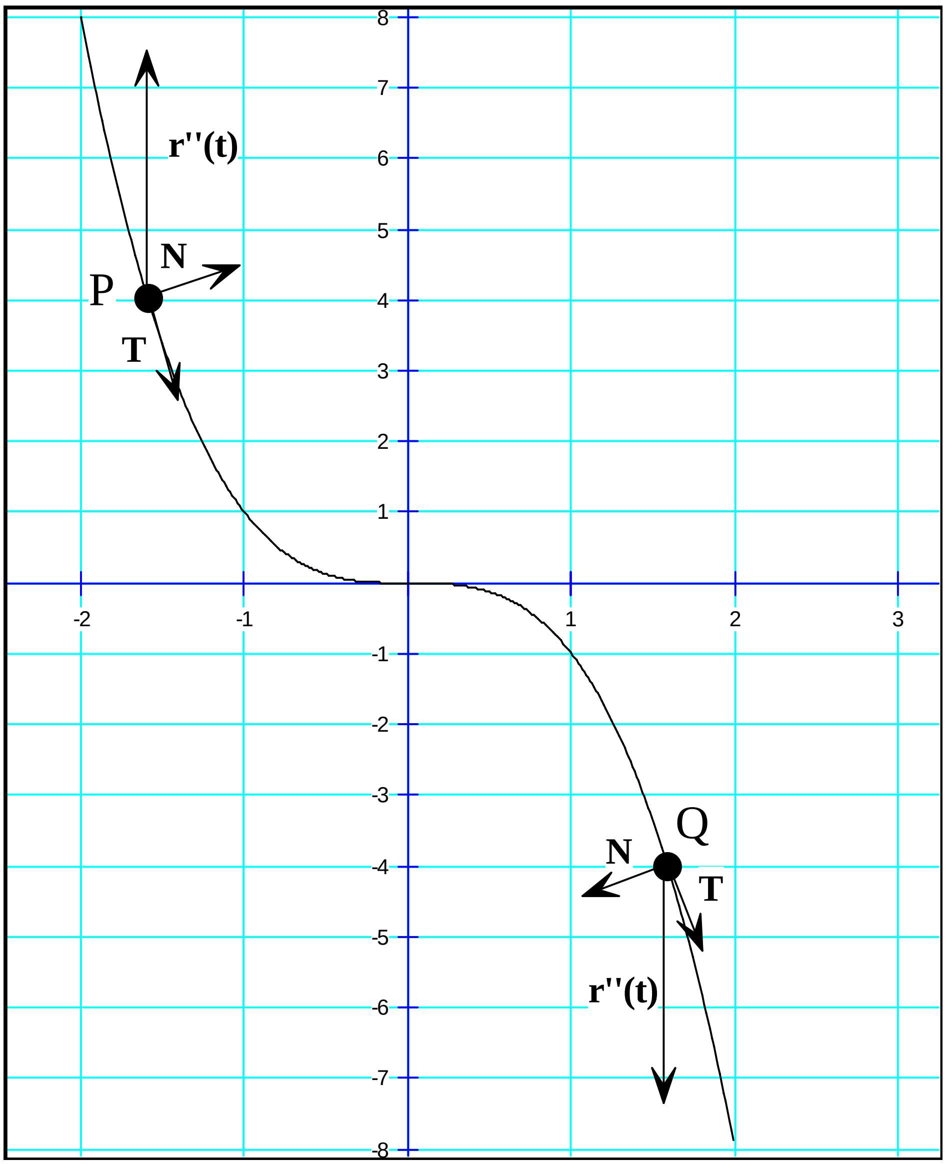
Our final comment about the nature of $\vec{r}''(t)$ has to do with its tangential and normal components. First, $k \geq 0$. In fact, unless C has a straight section, $k > 0$. So obviously kv^2 is normally positive when it isn't zero. Note that in order to have $v = 0$ we would have to have $\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} = 0$ and this only happens when all three first derivatives are simultaneously 0. In order for the three derivatives to be 0 over an interval, we would have to have $f(t),\ g(t),$ and $h(t)$ all be constants over that same interval and when that is the case then $\vec{r}(t)$ isn't much of a curve because it degenerates to a single constant point. So we can normally expect that $v = \left|\left|\vec{r}'(t)\right|\right|$ is positive. In fact many theorems dealing with space curves assume that $\left|\left|\vec{r}'(t)\right|\right| \neq 0$ for all $t \in [a, b]$. This can be stated more elegantly by requiring that the three derivative coordinate functions never vanish simultaneously at any point on the curve C. Another way of saying the same thing is that $\vec{r}'(t) \neq \vec{0}$ for all t. So normally the normal component of acceleration kv^2 is positive as long as $k > 0$ and C has no straight sections.

Now v being positive doesn't necessarily mean the tangential component $\dfrac{dv}{dt}$ must be positive or even nonzero. In fact, only when $\vec{r}'(t) \cdot \vec{r}''(t) < 0$ will $\dfrac{dv}{dt}$ be negative. In the above figure, when both components of acceleration are positive then the vector $\vec{r}''(t)$ will lie on the concave side of the curve C.

Now consider the example curve $\vec{r}(t) = \ < t, -t^3, 0 >$ were $t \in [-2, 2]$. This is really just the graph of $y = -x^3$ for $x \in [-2, 2]$ that has been parametrized. Then $\vec{r}'(t) = \ < 1, -3t^2, 0 > \ \neq \vec{0}$ and $\vec{r}''(t) = \ < 0, -6t, 0 >$. Then $\vec{r}'(t) \cdot \vec{r}''(t) = 18t^3$ which is negative when t is negative. In this case $\dfrac{dv}{dt} = \dfrac{18t^3}{\sqrt{1 + 9t^4}}$ which will also be negative when t is negative. We can easily calculate that $\vec{r}'(t) \times \vec{r}''(t) = \ < 0, 0, -6t >$ so that curvature as a function of time is given by $k(t) = \dfrac{\sqrt{36t^2}}{\left(\sqrt{1 + 9t^4}\right)^3} = \dfrac{6|t|}{\left(\sqrt{1 + 9t^4}\right)^3}$. Obviously $k = 0$ only where $t = 0$ which is at the origin point on C.

If we wanted to find the two points on the curve where the curvature $k(t)$ is a maximum we could do so by applying the quotient rule to the expression for $k(t)$. To make the calculation of $k'(t)$ a little simpler we might assume $t > 0$ so that $k(t) = \dfrac{6t}{\left(\sqrt{1 + 9t^4}\right)^3}$. By the symmetry of the graph shown on the next page we can expect to find two points on the curve where the curvature is the same maximum.

The following figure shows two different points on the curve where we can see the acceleration vector and its relation to the $\vec{T}$ and $\vec{N}$ vectors. The second quadrant point P is where $\dfrac{dv}{dt} < 0$. The fourth quadrant point Q is where $\dfrac{dv}{dt} > 0$. Also note that the normal vector $\vec{N}$ changes the side of the curve it is on when we move from P to Q. The tangent vector $\vec{T}$ almost always points down because its y-component is almost always negative. The only exception is when $t = 0$ and then $\vec{T} = \ < 1, 0, 0 >$ $= \vec{r}\,'(0)$ and then the tangent vector points to the right. Think of what happens to the vectors $\vec{T}$ and $\vec{N}$ and $\vec{r}\,''(t)$ at the origin point where $t = 0$. Those vectors are not shown at the origin point in the following figure.



For this example curve, $\vec{T}(t) = \dfrac{< 1, -3t^2, 0 >}{\sqrt{1+9t^4}}$ and $\vec{T}\,'(t) = \dfrac{< -18t^3, -6t, 0 >}{\left(\sqrt{1+9t^4}\right)^3}$. Then

$$\left|\left|\vec{T}\,'(t)\right|\right| = \frac{1}{\left(\sqrt{1+9t^4}\right)^3} \cdot || < -18t^3, -6t, 0 > || = \frac{1}{\left(\sqrt{1+9t^4}\right)^3} \cdot \sqrt{324t^6 + 36t^2} =$$

$\dfrac{1}{\left(\sqrt{1+9t^4}\right)^3} \cdot 6|t|\sqrt{9t^4+1} = \dfrac{6|t|}{1+9t^4}$. So $\left|\left|\vec{T}\,'(0)\right|\right| = 0$ and this means $\vec{N}$ will be undefined at the origin. However, we can calculate that when $t \neq 0$ then $\vec{N}(t) = \dfrac{1+9t^4}{6|t|} \cdot \dfrac{< -18t^3, -6t, 0 >}{\left(\sqrt{1+9t^4}\right)^3} =$ $\dfrac{-1}{|t|\sqrt{1+9t^4}} \cdot < 3t^3, t, 0 >$. If we take two one sided-limits we will find that even though $\vec{N}(0)$ is undefined, $\lim\limits_{t \to 0^-} \vec{N}(t) = < 0, 1, 0 >$ and $\lim\limits_{t \to 0^+} \vec{N}(t) = < 0, -1, 0 >$. These limit calculations imply that even though the tangent vector makes a smooth transition across the origin point, the normal vector makes an abrupt non-smooth change across that same point. Also, $\vec{T}(0) = < 1, 0, 0 >$ which means the tangent vector points along the positive x-axis. At the origin point we also have $\vec{r}\,''(0) =$ $< 0, 0, 0 >$ which means the acceleration vector vanishes at the origin.

If you try to maximize curvature $k(t)$ you should find $k'(t) = \dfrac{-270t^4 + 6}{\sqrt{1+9t^4}}$.
Setting the numerator equal to 0 and using the symmetry of the graph we estimate the two points on C where the curvature is a maximum are $(-0.386097395096,\ 0.0575560014247)$ and $(0.386097395096,\ -0.0575560014247)$ where $t = \pm \sqrt[4]{\dfrac{1}{45}} \approx \pm 0.386097395096$.